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Inequality for secant.
https://www.linkedin.com/feed/update/urn:li:activity:6614929031601229825
In a triangle ABC right angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that $5/2 < AB/BC < 3$.
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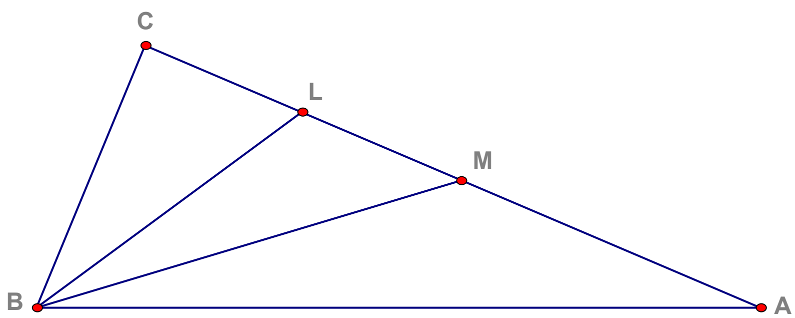

Since inequality $5/2 < AB/BC < 3$ is about the ratio of side lengths we can assume $BC = 1$.

Let $\varphi := \angle ABM = \angle LBM$. Then $\angle LBC = 2\varphi$, $AB = \dfrac{1}{\cos 4\varphi}$, $LC = \tan 2\varphi$, $MC = \tan 3\varphi$ and $AC = 2MC \Leftrightarrow \tan 4\varphi = 2\tan 3\varphi \Leftrightarrow \sin 4\varphi \cdot \cos 3\varphi - \cos 4\varphi \cdot \sin 3\varphi = \sin 3\varphi \cdot \cos 4\varphi \Leftrightarrow$ $\sin\varphi = \sin 3\varphi \cdot \cos 4\varphi \Leftrightarrow 1 = (3 - 4\sin^2\varphi)\cos 4\varphi \Leftrightarrow \cos 4\varphi = \dfrac{1}{3 - 4\sin^2\varphi} = \dfrac{1}{1 + 2\cos 2\varphi}$.

Let $t := 1 + 2\cos 2\varphi$. Then $\cos 4\varphi = \dfrac{1}{1 + 2\cos 2\varphi} \Leftrightarrow (2\cos^2 2\varphi - 1)(1 + 2\cos 2\varphi) - 1 = 0 \Leftrightarrow$ $\left((2\cos 2\varphi)^2 - 2\right)(1 + 2\cos 2\varphi) - 2 = 0$ becomes $\left((t-1)^2 - 2\right)t - 2 = 0 \Leftrightarrow$

(1) $\quad t^3 - 2t^2 - t - 2 = 0$.

Let $h(t) := t^3 - 2t^2 - t - 2$. Since cubic equation (**1**) has single real positive root t_0 and $h\left(\dfrac{5}{2}\right) < 0, h(3) > 0$ then $\dfrac{5}{2} < t_0 < 3$. Hence, noting that $\dfrac{AB}{BC} = \dfrac{1}{\cos 4\varphi} = 1 + 2\cos 2\varphi = \dfrac{1}{t_0}$, we obtain $\dfrac{5}{2} < \dfrac{AB}{BC} < 3$.

Remark.

More weak Inequality $\sqrt{2} + 1 < \dfrac{AB}{BC} < 3$ (because $\sqrt{2} + 1 < \dfrac{5}{2}$) can be obtained without finding φ, namely $\dfrac{1}{\cos 4\varphi} = 3 - 4\sin^2\varphi < 3$ and $4\varphi < \dfrac{\pi}{2} \Leftrightarrow \varphi < \dfrac{\pi}{8}$ implies

$3 - 4\sin^2\varphi > 3 - 4\sin^2\dfrac{\pi}{8} = 3 - 2\left(1 - \cos\dfrac{\pi}{4}\right) = \sqrt{2} + 1$.